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ANALYTIC VALUE FUNCTION IN MULTI-OBJECTIVE OPTIMIZATION

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Abstract

The present study takes a new approach to the classical problem of multi-objective optimization of finding a suitable value function (also known as utility function). Instead of dealing with real valued value functions, the implications of a complex-valued value function (defined in a suitable way), which is also analytic are discussed. The classical results that are unique to analytic functions yield numerous interesting insights in this relation.

1. Introduction

One of the most well-known problems in multi-objective optimization is the study of a real-valued value function on a subset X of the Euclidean space $\mathbf{R}^{\mathbf{n}}$, on which a complete preference structure is defined. This is a function, which assigns to each vector of the set X a real number in such a way that the preference structure is preserved.

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The biggest advantage of a real-valued value function is that, it practically reduces the multi-objective optimization problem to a single objective optimization problem, because all we need to do is to optimize the value function, which is real-valued and hence a one-dimensional function. There is an enormous amount of literature about methods for solving single-objective optimization problems. There are exact as well as approximate and heuristic algorithms for solving such problems (See [4], [5], and [17], for instance).

Researchers have worked on finding sufficient conditions for a continuous value function. For example, Debreu [9] has shown that under the very reasonable assumptions of continuity of the preference relation (to be described in more detail in Section 3), there exists a continuous value function. Similarly, work has been done on conditions for a differentiable value function. For example, Atakan [2] has explored sufficient conditions for a concave and differentiable value function in dynamic programming problems.

The advantage of a differentiable value function is that, we can freely use calculus in our analysis. But, if we could have an analytic value function, then we would have a host of other useful information and analytical tools at our disposal because of the additional structures which such functions possess.

2. The Classical Value Functions

In multi-objective optimization, one attempts to optimize multiple, oftentimes conflicting objectives at the same time. It is generally not possible to optimize all these objectives simultaneously. One therefore tries to optimize in some way the overall problem. We have to make tradeoffs in order to do that. The classical multi-objective optimization problem is thus defined as follows:

Let X be the set of feasible alternatives in a given experiment. If there are n decision variables, then each alternative can be regarded as a point of $\mathbf{R^n}$. Thus X will be a subset of $\mathbf{R^n}$. We will call X, the alternatives space. If there are m objectives, then with each $x \varepsilon X$, we associate an element of $\mathbf{R^m}$. The set Y of all possible images of points in X will be called the *target space* and it is assumed to have a preference structure defined on it. This means that given $u, \nu \varepsilon Y$, one and only one of the following holds:

(i) u is preferred to ν, written as $u \succ \nu$.

(ii) ν is preferred to u, written as $\nu \succ u$.

(iii) u and ν are indifferent, written as $u \sim \nu$.

If $m \geq 2$, we cannot have an optimum solution to this problem in the classical sense of the word. Instead, we talk about efficient points (see [6] and [13], for instance). Different methods are available to deal with such problems (see [10], [11], [15], and [16], for instance). One very common and popular one is to search for a value function on the target space Y. By this, we mean to find a real valued function $f : Y \to \mathbf{R}$ that has the following property:

$f(u) > f(\nu)$, if and only if $u \succ \nu$, and $f(u) = f(\nu)$, if $u \sim \nu$.

Once a value function has been found the problem, then reduces to optimizing this value function.

Let us point out here that not all subsets of $\mathbf{R^n}$ with a given preference structure can have a value function defined on them, for a counter example, see notes at the end of Chapter 4 of [9].

It is also shown in [9] that, if Y is a connected subset $\mathbf{R^m}$ with a preference structure defined on it, then there exists a continuous utility function on Y, if the following condition holds:

For every $w \in Y$, the set of all those elements of Y that are preferred to or indifferent to w is a closed subset of Y, and also the set of all those elements of Y to which w is preferred or indifferent is a closed subset of Y.

3. Implications of Analytic Value Functions

A review of literature reveals that the only value functions that have been discussed till now are those which are real-valued. Some work has been done on linear programming problems with complex values ([7]), and on problems in a complex space setting ([8] and [12]), and even on vector space methods ([14]), but none on complex-valued value functions. Hence, since by a value function, we traditionally understand a real valued function, the term 'analytic value function' needs a precise definition.

Definition. Let Y be a subset of $\mathbf{C}$ with a preference structure defined on it. An analytic function, $f : Y \to \mathbf{C}$ is said to be an analytic value function, if $|f(z)| > |f(w)|$, whenever $z \succ w$, and $|f(z)| = |f(w)|$, whenever $z \sim w$.

First, we note that we have stated this definition only for subsets of $\mathbf{C}$, but this can be generalized to subsets of $\mathbf{C^n}$, the n-dimensional complex space. The one dimensional complex space is essentially the same as the space $\mathbf{R^2}$, while the complex space of dimension $n > 1$ is analogous to the real space of dimension $2n$. It may thus seem that by discussing complex analytic functions, we will essentially be treating only even dimensional real spaces but in practical problems, we can usually overcome this difficulty by adding a dummy variable, if needed to have an even number of variables. In the present article, we will restrict ourselves only to analytic value functions over one dimensional complex spaces.

We will talk about the existence of an analytic value function in the next section. For now, let us examine the implications, if such a value function were to exist. First of all, we know that an analytic function is infinitely differentiable. This is very useful because we can freely use calculus with such functions. Whereas, continuity ensures that there are no breaks in the graph, differentiability ensures smoothness, which is also a useful property.

Next, the classical Cauchy integral formula tells us that, if a function is analytic on and within the boundary of a region, and we know its values on the boundary, we can calculate its values at any interior point. It is well-known that many a times, we cannot completely find an explicit continuous value function, but we know that one does exist. In such situations, we sometimes know some other properties of the function (for example, convexity) and perhaps its values at some points, and even this apparently small knowledge helps us draw certain useful conclusions.

Thus, if we ever run into a similar situation with an analytic value function not known explicitly Cauchy integral formula tells us that the knowledge of just the boundary values is enough to know all the values. We know that even for infinitely differentiable real-valued functions, a knowledge of the boundary values gives us no clue about the values in the interior.

But, the most stunning complex analysis fact in relation to this situation is the maximum principle (see [1], for instance), which implies that a mere knowledge of the boundary values is enough! We know that a continuous function on a compact set attains a maximum. Obviously, an analytic function is continuous on its domain of definition. If a function is analytic on a bounded domain Ω and is continuous up to the boundary, then it will have a maximum on the closure of Ω. Hence, maximum principle implies that a nonconstant analytic function, that is, continuous up to the boundary has its maximum on the boundary, since it cannot have a maximum on any interior point. This reminds us of linear programming, where we know that the optimum occurs at corner points of the feasible region. Thus, even if we cannot exactly maximize an analytic value function, the knowledge that the maximum occurs at the boundary can be very helpful.

Finally, we know that if the values of two analytic functions agree on a certain 'very small' countable set, namely, on any sequence of points, which has a limit point in the domain, then they agree everywhere in the domain of definition. Clearly, all constant functions are analytic. Thus, if the function has the same values on a certain extremely small region, then we know that it is constant everywhere. This, however, is unlikely to happen in most practical problems, although sometimes, if target space is contained within a very small ball and we are able to find an analytic value function, then this property might come into play.

4. Necessary and Sufficient Conditions

In this section, we will give necessary and sufficient conditions for the existence of an analytic value function. We will also describe relationships between analytic value functions and the classical real-valued functions.

Theorem 4.1. *If f is an analytic value function on a subset Y of $\boldsymbol{C}$, then the preference structure on Y must be transitive.*

Proof. By a transitive preference structure, we mean that if a, b, c are elements of Y such that $a \succ\sim b$, $b \succ\sim c$, then $a \succ\sim c$. We will prove by contradiction. Suppose the preference structure is not transitive. Thus, there exist complex numbers a, b, c such that $a \succ\sim b$, $b \succ\sim c$, but $c \succ a$.

Now, $a \succ\sim b \Rightarrow |f(a)| \geq |f(b)|$,

and $b \succ\sim c \Rightarrow |f(b)| \geq |f(c)|$.

The above two relations then imply that $|f(a)| \geq |f(c)|$. But according to our assumption,

we have $c \succ a \Rightarrow |f(c)| > |f(a)|$.

This is a contradiction and completes our proof. □

Next, we shall state and prove two theorems that relate continuous real-valued value functions to analytic value functions. An interesting question is this: Given a continuous real-valued value function f on a subset Y of $\mathbf{C}$ (regarded as a subset of $\mathbf{R}^2$), when does there exist an analytic function g on Y with the property that $|g| = f$ on Y? By definition, such a g, if it exists, will be an analytic value function. The next two theorems shed light on this question.

Theorem 4.2. *Suppose there exists a continuous, positive, and real-valued value function f on a subset Y of $\boldsymbol{C}$ (regarded as a subset of $\mathbf{R}^2$). Suppose there is also an analytic value function g on Y with the same preference structure, such that $|g| = f$ on Y. Then,* ln *f must be harmonic, where* ln *denotes the natural logarithm.*

Proof. Suppose the hypothesis of the theorem is satisfied. So,

$$|g| = f,$$

$$\Rightarrow \ln|g| = \ln f.$$

Note that the assumption that f be positive allows us to take its natural logarithm. We claim that, if g is analytic, then $\ln|g|$ is harmonic. This can be proved as follows:

Let $g(x,\ y) = u(x,\ y) + i\nu(x,\ y)$

$$\Rightarrow h := \ln|g| = \frac{1}{2}\ln(u^2 + \nu^2)$$

$$\Rightarrow h_{xx} = \frac{(u^2 + \nu^2)(uu_{xx} + u_x{}^2 + \nu\nu_{xx} + \nu_x{}^2) - (uu_x + \nu\nu_x)(2uu_x + 2\nu\nu_x)}{(u^2 + \nu^2)^2},$$

and

$$h_{yy} = \frac{(u^2 + \nu^2)(uu_{yy} + u_y{}^2 + \nu\nu_{yy} + \nu_y{}^2) - (uu_y + \nu\nu_y)(2uu_y + 2\nu\nu_y)}{(u^2 + \nu^2)^2}.$$

Adding the last two equations, and using the fact that $u_x\nu_x + u_y\nu_y = 0$ because of the Cauchy-Riemann equations we have, after simplification:

$$h_{xx} + h_{yy} = \frac{(u^2 - \nu^2)(-u_x{}^2 + \nu_x{}^2 - u_y{}^2 + \nu_y{}^2)}{(u^2 + \nu^2)^2}. \tag{5.1}$$

By Theorem 2.7, we have

$$|g'(z)|^2 = u_x{}^2 + u_y{}^2 = \nu_x{}^2 + \nu_y{}^2,$$

$$\Rightarrow \nu_x{}^2 + \nu_y{}^2 - u_x{}^2 - u_y{}^2 = 0.$$

This means that the value of the second parenthesis in the numerator of (5.1) is 0. So,

$$h_{xx} + h_{yy} = 0.$$

Thus h is harmonic, which means that $\ln|g|$ is harmonic and hence $\ln f$ is harmonic. □

The converse of the above theorem is also true, namely:

Theorem 4.3. *Suppose there exists a continuous, positive, and real-valued value function f on a simply connected domain Ω (regarded as a subset of $\mathbf{R^2}$), such that $\ln f$ is harmonic in Ω. Then, there exists an analytic value function g on Ω such that $|g| = f$ on Ω.*

Proof. By a well-known theorem in complex analysis, if a function is harmonic in a simply connected domain, then it is the real part of an analytic function in that domain. See [5], page 201, for a proof of this fact. Thus, under the hypothesis of the theorem, there exists an analytic function h on Ω such that $\ln f = \operatorname{Re} h$, where the notation used on the right hand side means the real part of h.

Define a function g on Ω as follows:

$$g(z) = \exp(\operatorname{Re} h + i \operatorname{Im} h) = \exp(h).$$

Since, the exponential of an analytic function is also analytic, the analyticity of h implies the analyticity of g. Also, it is easy to see that the absolute value of an exponential function is equal to the exponential of its real part. Thus,

$$\begin{aligned} |g| &= \exp(\operatorname{Re} h) \\ &= \exp(\ln f) \\ &= f. \end{aligned}$$

This proves the theorem. □

Combining the statements of Theorems 5.3 and 5.4, we get our big result:

Theorem 4.4. *Suppose there exists a continuous, positive, and real-valued value function f on a simply connected domain Ω. Then a necessary and sufficient condition for the existence of an analytic value function g on Ω such that $|g| = f$ on Ω is that* $\ln f$ *be harmonic in* Ω.

5. Conclusions

We have defined and explored a new concept, that of an analytic value function. We have described its implications and discussed necessary and sufficient existence conditions. There are a host of analytic functions and their exploration as value functions in different situations that could be a direction of future research in this area.

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